Li-Yau Harnack inequality in a manifold with a non-convex boundary

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Outline

1. Motivation
2. Manifold without boundary
3. Manifold with boundary

Motivation

Theorem (Haslhofer-Kopfer-Naber 20)

If $Ric = 0$*, then the differential harnack inequality on path space holds: For each* $F \geqslant 0$

$$\frac{\mathbb{E}_x(D_h D_h F)}{\mathbb{E}_x(F)} - \frac{|\mathbb{E}_x(D_h F)|^2}{\mathbb{E}_x(F)^2} + \|h\|^2 \geqslant 0$$

which implies that the Li-Yau harnack inequality on M*:* $Ric = 0$,

$$\frac{\Delta f_t}{f_t} - \frac{|\nabla f_t|^2}{f_t^2} + \frac{n}{2t} \geqslant 0.$$

Remark:

The proof is based on the integration by parts formula and the divegence theorem on path space;

Motivation

Aim:

Consider the differential Harnack inequality on the path space over a manifold with a nonconvex boundary, and the Li-Yau Harnack inequality in the associated manifold.

Idea:

Based on the integration by parts formula and the convexity property of the functional Ψ;

Manifold without boundary

Let M be a complete Riemannian maniflod.

- Heat Equation: $\partial_t u_t = \Delta u_t$;
- Li-Yau Harnack Inequality: Ric $\geqslant 0$,

$$\frac{\Delta u_t}{u_t} - \frac{|\nabla u_t|^2}{u_t^2} + \frac{n}{2t} \geqslant 0;$$

- Standard Harnack Inequality:

$$u_t(x) \leqslant (4\pi(t-s))^{\frac{n}{2}} e^{\frac{d(x,y)^2}{4(t-s)}} u_t(y);$$

- Wang's Harnack Inequality: Ric $\geqslant -K$. Let P_t = heat semigroup,

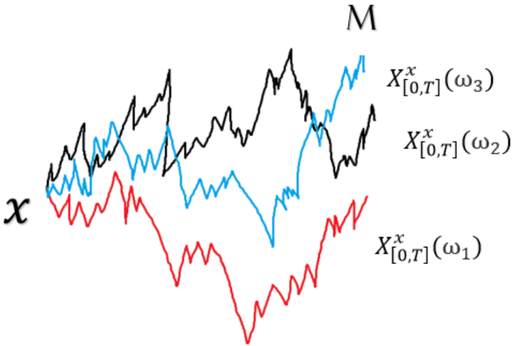
$$|P_t f(x)|^\alpha \leqslant |P_t|^\alpha f(y) \exp\left[\frac{\alpha C \rho(x,y)^2 \int_0^t g(s)ds}{4(\alpha-1)\left(\int_0^t g(s)e^{-Ks}ds\right)^2}\right];$$

- Hamilton's Matrix Harnack Inequality : Sec $= 0$ and ∇ Ric $= 0$

$$\frac{\nabla_i \nabla_j u_t}{u_t} - \frac{\nabla_i u_t \cdot \nabla_j u_t}{u_t^2} + \frac{g_{i,j}}{2t} \geqslant 0.$$

Brownian Motion on M:

M: Riemannian manifold, fix $x \in M$ and $T > 0$



Notation

- $W_x(M) := \{\gamma \in C([0,1];M); \gamma(0) = x\}$;
- Wiener Measure μ_x: The law of Brownian motion $X^x_{[0,1]}$;
- $U_\cdot(\gamma)$: Stochastic horizontal lift along $\gamma(\cdot)$;

$$dU_t = \sum_{i=1}^n H_i(U_t) \circ dB_t^i, \quad U_0 = u \in O(M);$$

- Cylinder functions: $\mathscr{F}C_b^{L^2}, \mathscr{F}C_b^{OU}$

$$\mathscr{F}C_b^{L^2} = \left\{F(\gamma) = f\left(\int_0^{t_1} g_1(s,\gamma_s)ds, \cdots, \int_0^{t_n} g_n(s,\gamma_s)ds\right)\right\}$$

$$\mathscr{F}C_b^{OU} = \left\{F(\gamma) = f(\gamma_{t_1}, \cdots, \gamma_{t_n}) : 0 < t_1 < \cdots < t_n \leqslant 1\right\}.$$

Notation

- Cameron-Martin space:

$$\mathbb{H} := \left\{ h \in C([0,1];\mathbb{R}^d) \Big| h(0)=0, \int_0^1 |\dot h(s)|^2 ds < \infty \right\};$$

- Direction derivative:

$$D_hF(\gamma) = \frac{d}{d\varepsilon} F(\exp_\gamma \varepsilon Uh)\Big|_{\varepsilon=0}, \quad h \in L^2;$$

- Malliavian gradient and L^2-gradient:

$$D_hF(\gamma) = \langle D^O F(\gamma), h\rangle_{\mathbb{H}}, \quad h \in \mathbb{H}$$

$$D_{\tilde h}F(\gamma) = \langle D^{L^2} F(\gamma), \tilde h\rangle_{\mathbb{H}_0}, \quad \tilde h \in L^2;$$

Riemannian path space

- O-U Dirichlet form: $F \in \mathscr{F}C_b^{OU}$.

$$\mathscr{E}^{OU}(F,F) := \int \int_0^1 \left| \frac{d}{ds} D^O F(s) \right|^2 \mathrm{d}s \; \mathrm{d}\mu \longleftrightarrow \text{O-U Process}$$

- L^2-Dirichlet form: $F \in \mathscr{F}C_b^{L^2}$

$$\mathscr{E}^{L^2}(F,F) = \int \int_0^1 \left| D^{L^2} F(s) \right|^2 \mathrm{d}s\mathrm{d}\mu \longleftrightarrow \text{Stoch. Heat Equ..}$$

Notation

- The differential of associated Itô map:

  [J.M. Bismut 84], [S.Z. Fang, P. Malliavin 93], [A.B. Cruzeiro, P. Malliavin 96], [X.D. Li, T. Lyons 06], [K.D. Elworthy, X.M. Li 07];

- Existence of a quasi-invariance flow:

  [B. Driver 92], [E. Hsu 95], [X.D. Li 04], [E. Hsu 02], [F.Z. Gong, J.X. Zhang 07], [E. Hsu, C. Ouyang 09],[X. Chen, X. M. Li, W. 21+];

- Quasi-regular O-U Dirichlet form: (1) Compact[B. Driver, M. Röckner 92],[S. Z. Fang, P.Malliavin 93], [O. Enchev, D. W. Stroock 95], [A. B. Cruzeiro, P. Malliavin 96], [D.Elworthy, Z. M. Ma 97], [D.Elworthy, X. M. Li, Y. Lejan 99], [Löbus 04] (2) Noncompact [F. Y. Wang, W. 08], [X. Chen, W. 14];

Known Results

- Quasi-regular L^2-Dirichlet form: (1) Compact[M. Röckner, W., R. C. Zhu, X. C. Zhu 20]; (2)Noncompact [X. Chen, W., R. C. Zhu, X. C. Zhu 21];
- Uniqueness of the stochastic heat equation: [Y. Bruned, F. Gabriel, M. Hairer, L. Zambotti 21];
- Differential geometry on $W_x(M)$:

  [A.B. Cruzeiro, P. Malliavin 96], [S.Z. Fang, A.B. Cruzeiro 97,01], [S.Z. Fang 99], [X.D. Li, 00,03], [K.D. Elworthy, X.M. Li 06];
- Finite dimensional approximation: [L. Andersson, B. Driver 99], [A.B. Cruzeiro, X.C. Zhang 02], [W. 18];

Known Results

- Characterization of two-sided bounds of Ricci curvature:

  [A. Naber 13], [S.Z. Fang, W. 17], [F.Y. Wang, W. 18], [L. Cheng, A. Thalmaier 17], [W. 18,19];

- Functional inequalities

  O-U Dirichelt Form: [S.Z. Fang 94], [E. Hsu 97], [M. Hino 98], [F.Y. Wang 02], [S.Z. Fang, F.Y. Wang 04], [S.Z. Fang, S.Shao 05], [K.D. Elworthy, Y. LeJan, X.M. Li 07], [X. Chen, W. 14], [W. 19+];

  L^2-Dirichelt Form: [M. Gorcy, L. Wu 07], [M. Röckner, W., R.C. Zhu, X.C. Zhu 20],[X. Chen, W., R.C. Zhu, X.C. Zhu 21];

  TCI: [S.Z. Fang, J.H. Shao 05], [J.H. Shao 06], [S.Z. Fang, F.Y. Wang, W. 08], [F.Y. Wang 14].

Differential Harnack Inequality on $W_x(M)$

Notation:

- φ-gradient: $\nabla_\varphi F : W_x(M) \to T_x M$

  $\langle \nabla_\varphi F(\gamma), V \rangle = D_{V_\varphi} F(\gamma), \quad V \in T_x M, \gamma \in W_x(M), V_\varphi = \varphi U_0^{-1} V.$

- Cruzeiro-Malliavin's Markovian Connection:

  $\mathbf{\frac{d}{dt} U_t^{-1}(\gamma)(\nabla_V W)_t = D_V \dot{w}_t + \int_0^t U_s R_{\gamma_s}(V_s, \dot{\gamma}_s) ds \dot{w}_t},$

  where $\mathbf{v_t, w_t : W_x(M) \to \mathbb{R}^d}$ and $\mathbf{V = U_t v_t, W = U_t w_t}$.

- φ-Hessian: $\mathbf{Hess_\varphi F : W_x(M) \to T_x^* M \times T_x^* M}$

  $\mathbf{Hess_\varphi F(v, v) = Hess F(V_\varphi, V_\varphi)},$

  where $\mathbf{Hess F(V, W) = D_v D_w F - D_{U_t^{-1}(\gamma)(\nabla_V W)} F}$.

- φ-Laplacian: $\mathbf{\Delta_\varphi F = tr(Hess_\varphi F) : W_x(M) \to \mathbb{R}}$

Theorem (Haslhofer-Kopfer-Naber 20)

Differential harnack inequality holds: For each $F \geqslant 0$

$$\frac{\mathbb{E}_x(\Delta_\varphi F)}{\mathbb{E}_x(F)} - \frac{|\mathbb{E}_x(\nabla_\varphi F)|^2}{\mathbb{E}_x(F)^2} + \Big(\frac{n}{2} + C(Ric) + C(R, \nabla Ric)\frac{\mathbb{E}_x(F^2)^{1/2}}{\mathbb{E}_x(F)}\Big)\|\varphi\|^2 \geqslant 0$$

In particular, suppose that $Ric = 0$,

$$\frac{\mathbb{E}_x(\Delta_\varphi F)}{\mathbb{E}_x(F)} - \frac{|\mathbb{E}_x(\nabla_\varphi F)|^2}{\mathbb{E}_x(F)} + \frac{n}{2}\|\varphi\|^2 \geqslant 0,$$

which implies that the Li-Yau harnack inequality: $Ric = 0$,

$$\frac{\Delta f_t}{f_t} - \frac{|\nabla f_t|^2}{f_t^2} + \frac{n}{2t} \geqslant 0.$$

Remark The proof is based on the integration by parts formula and the divegence theorem on path space;

Differential Harnack Inequality on $W_x(M)$

Notation:

- Orthonormal basis of $\mathbb{H}$: $h_n \in \mathbb{H}$ (Haar fuct.)
- O-U Dirichelt form: Assume that $\lambda_n \geqslant \delta I$ for some $\delta > 0$.

$$\mathscr{E}_A(F,F) = \int_{W(M)} \langle DF, ADF \rangle_{\mathbb{H}} d\mu$$

$$A\phi(\gamma) := \sum_{n=1}^{\infty} \lambda_n \langle h, h_n \rangle_{\mathbb{H}} h_n, \quad \phi \in L^2(W_x(M) \to \mathbb{H}; \mu_x);$$

- $Y \subset \mathscr{F}C_b^{OU}$: $Y := \left\{ F(\gamma) = f(\gamma_{s_1}, \cdots, \gamma_{s_m}), s_i \in N_0 \right\}$, where

$$N_0 = \left\{ \frac{l}{2^n} : l \in 1, 2, \ldots, 2^n, n \in \mathbb{N} \right\}.$$

Differential Harnack Inequality for $h \in \mathbb{H}$

Theorem

$$\frac{\mathbb{E}\left[D_h D_h F\right]}{\mathbb{E}\left[F\right]} - \frac{\left[\mathbb{E}(D_h F)\right]^2}{\mathbb{E}\left[F\right]^2} + \frac{1}{2} A_F \geqslant 0, \quad h \in \mathbb{H}.$$

Which implies that the Li-Yau harnack inequality: $Ric = 0$,

$$\frac{\Delta f_t}{f_t} - \frac{|\nabla f_t|^2}{f_t^2} + \frac{n}{2t} \geqslant 0.$$

Idea of the proof

Convexity of the functional: $\frac{d^2}{d\varepsilon^2}\Psi_{F,h}(\varepsilon)\Big|_{\varepsilon=0} \geqslant 0$:

$$\Psi_{F,h}(\varepsilon) := \log \mathbb{E}\left[F\left(\xi^{\varepsilon,h}_{[0,1]}\right)\right] = \log\left(\int_\Omega F\left(\xi^{\varepsilon,h}_{[0,1]}\right)\, d\mathbb{P}\right) + \frac{1}{2}\varepsilon^2 A_F, \quad \varepsilon \geqslant 0.$$

where $\xi^{\varepsilon,h}_{[0,1]}$ is the quasi-invarant flows of μ_x and

$$A_F := \frac{\mathbb{E}\left[F\left(X^x_{[0,T]}\right)\left\{\left(\frac{d}{d\varepsilon}Q^{\xi^{\varepsilon,h}}\right)^2\Big|_{\varepsilon=0} - \frac{d^2}{d\varepsilon^2}Q^{\xi^{\varepsilon,h}}\Big|_{\varepsilon=0}\right\}\right]}{\mathbb{E}\left[F\left(X^x_{[0,1]}\right)\right]}$$

$$\int_\Omega F(\xi^{\varepsilon,h}_{[0,1]})d\mathbb{P} = \int_\Omega F(X_{[0,1]})Q^{\xi^{\varepsilon,h}}d\mathbb{P}.$$

Idea of the proof

Remark:

Our method may be applied to more general cases, such as Riemannian loop spaces and fractional Wiener spaces.

Lemma (Löbus 04)

The generator of $\mathscr{E}_A$:

$$\mathscr{L}_A F(\gamma) = \sum_{n=1}^{\infty} \lambda_n \{D_{h_n} D_{h_n} F + \Theta_n D_{h_n} F\},$$

where $\Theta_n = \int_0^1 \left\langle \frac{1}{2} h_n'(t) + \frac{1}{2} Ric_{U_t}(h_n(t)), dB_t \right\rangle$

Theorem

For each nonnegative $F \in Y$*, there exists* $N \in \mathbb{N}$ *such that*

$$\frac{\mathbb{E}\left[\mathscr{L}_A^0 F\right]}{\mathbb{E}\left[F\right]} - \frac{\left|\mathbb{E}\left[A^{1/2} F\right]\right|^2}{\mathbb{E}\left[F\right]^2} + \sum_{m=1}^{N} \lambda_m \frac{\mathbb{E}\left[F\left\{\frac{1}{2}\int_0^1 \langle \Lambda_s, dB_s\rangle + \frac{1}{2}\int_0^1 |\Theta_s|^2 \, ds - D_{h_m}^* \Theta_m\right\}\right]}{\mathbb{E}\left[F\right]} \geqslant 0,$$

where Λ_s *depends on* h, Ric *and* ∇Ric.

Corollary

Assume that $C = \sum_{m=1}^{\infty} \lambda_m < \infty$.

$$\frac{\mathbb{E}\left[\mathscr{L}_A F\right]}{\mathbb{E}\left[F\right]} - \frac{\left|\mathbb{E}\left[A^{1/2}\nabla F\right]\right|^2}{\mathbb{E}\left[F\right]^2} + \sum_{m=1}^{\infty} \lambda_m \frac{\mathbb{E}\left[F\left\{\frac{1}{2}\int_0^1 \langle \Lambda_s, dB_s\rangle + \frac{1}{2}\int_0^1 |\Theta_s|^2\, ds - D_{h_m}^* \Theta_m\right\}\right]}{\mathbb{E}\left[F\right]} \geqslant 0,$$

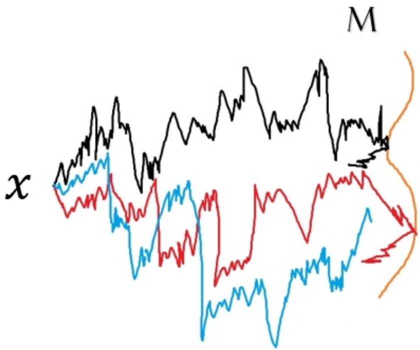
In particular, when $M = \mathbb{R}^d$, *we have*

$$\frac{\mathbb{E}\left[\mathscr{L}_A F\right]}{\mathbb{E}\left[F\right]} - \frac{\left|\mathbb{E}\left[A^{1/2}\nabla F\right]\right|^2}{\mathbb{E}\left[F\right]^2} + \frac{C}{2} - \sum_{m=1}^{\infty} \lambda_m \frac{\mathbb{E}\left[F D_{h_m}^* \Theta_m\right]}{\mathbb{E}\left[F\right]} \geqslant 0, \quad F \in \mathscr{F}C_b^{OU}, F$$

Manifold with boundary

Reflection diffusion process on M:

M: Riemannian manifold with boundary, fix $x \in M$ and $T > 0$



Wiener Measure μ_x: The law of the reflection diffusion process $X^x_{[0,1]}$.

Notation

Notation:

- Second fundamental form on ∂M:

$$\mathbb{I}_u(a,b) = \mathbb{I}(\pi_\partial ua, \pi_\partial ua), \quad a,b \in \mathbb{R}^n,$$

where $\pi_\partial : TM \to T\partial M$ is the orthogonal projection at points on ∂M.

- Adapted Cameraon-Martin space $\mathbb{H}_0$:

$$\mathbb{H}_0 = \left\{ h \in L^2(\Omega \to \mathbb{H}; \mathbb{P}) : h_t \text{ is } F_t - \text{measurable}, \ t \in [0,T] \right\}.$$

- Wang's damped quasi-invariant flows: For each $h \in \mathbb{H}_0$

$$\begin{cases} dX_t^x = \sqrt{2}\, U_t^x \circ dB_t + N(X_t) dl_t^x, & X_0^x = x, \\ dX_t^{\varepsilon,h} = \sqrt{2}\, U_t^{\varepsilon,h} \circ dW_t + N(X_t^{\varepsilon,h}) dl_t^{\varepsilon,h} + \sqrt{2}\varepsilon\, U_t^{\varepsilon,h} h_t' dt, & X_0^{\varepsilon,h} = x, \end{cases}$$

where N is the inward unit normal vector field of ∂M.

Notation

Notation:

- Damped gradient ∂M:

$$\frac{d}{dt}\tilde{D}F(X^x_{[0,1]})(t) = \sum_{i:t_i>t} Q^x_{t,t_i} U^{-1}_{t_i} \nabla_i f(X^x_{t_1}, \cdots, X^x_{t_N}), \quad t \in [0,1],$$

where

$$Q^x_{s,t} = \left(I - \int_s^t Q^x_{s,r}\left\{Ric(U^x_r)ds + \mathbb{I}(U^x_r)dl^x_r\right\}\right)\left(I - 1_{\{X^x_t \in \partial M\}} P_{U^x_t}\right),$$

here $P_u : \mathbb{R}^d \to \mathbb{R}^d$ is the projection along $u^{-1}N$, i.e.

$$\langle P_u a, b\rangle := \langle ua, N\rangle\langle ub, N\rangle, \quad a, b \in \mathbb{R}^d, u \in \cup_{x\in\partial M} O_x(M).$$

- Integration by parts formula(Wang):

$$\mathbb{E}(D^0_h F(X^x_{[0,1]})) := \langle \tilde{D}F(X^x_{[0,1]}), h\rangle_{\mathbb{H}} = \frac{\sqrt{2}}{2}\mathbb{E}\left(\frac{d}{d\varepsilon}F(X^{\varepsilon,h}_{[0,1]})\right)\Big|_{\varepsilon=0}$$

Differential Harnack Inequality on $W_x(M)$

Theorem

For each nonnegative $F \geqslant 0$,

$$\frac{\mathbb{E}\left[D_h^0 D_h^0 F\right]}{\mathbb{E}\left[F\right]} - \frac{|\mathbb{E}(D_h^0 F)|^2}{\mathbb{E}\left[F\right]^2} + \frac{1}{2}\|h\|_{\mathbb{H}}^2 \geqslant 0, \quad h \in \mathbb{H}_0.$$

Corollary

Assume that Ric $= 0$. Then we have

$$\frac{\Delta P_t f(x)}{P_t f(x)} - \frac{|\nabla P_t f(x)|^2}{P_t f(x)} + \frac{\alpha(t)}{2t} \geqslant 0,$$

where $\alpha(t) = \sum_{i=1}^n \mathbb{E}\left[\frac{1}{|A_t e^i|^2}\right]$ *and* $A_t := \frac{1}{t}\int_0^t U_t(Q_{s,t}^x)^* ds : \mathbb{R}^n \to T_{X_t}M$.

Differential Harnack Inequality on $W_x(M)$

Theorem

For each nonnegative $F \in Y$, there exists $N \in \mathbb{N}$ such that

$$\frac{\mathbb{E}\left[\mathscr{L}_A^0 F\right]}{\mathbb{E}\left[F\right]} - \frac{\left|\mathbb{E}\left[A^{1/2}\tilde{D}F\right]\right|^2}{\mathbb{E}\left[F\right]^2} + \sum_{m=1}^{N} \frac{\lambda_m}{2} \frac{\mathbb{E}\left[F\left\{1 - D_{h_m}^* \int_0^1 \left\langle h_m'(t), dB_t \right\rangle\right\}\right]}{\mathbb{E}\left[F\right]} \geqslant 0.$$

Corollary

Assume that $C = \sum_{m=1}^{\infty} \lambda_m < \infty$.

$$\frac{\mathbb{E}\left[\mathscr{L}_A^0 F\right]}{\mathbb{E}\left[F\right]} - \frac{\left|\mathbb{E}\left[A^{1/2}\tilde{D}F\right]\right|^2}{\mathbb{E}\left[F\right]^2} + \sum_{m=1}^{\infty} \frac{\lambda_m}{2} \frac{\mathbb{E}\left[F\left\{1 - D_{h_m}^* \int_0^1 \langle h_m'(t), dB_t \rangle\right\}\right]}{\mathbb{E}\left[F\right]} \geqslant 0, \quad F \in \mathscr{F}C_b^{\infty}, F \geqslant 0.$$

Thanks for your attention!